

## Continuum time limit and stationary states of the minority game

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We discuss in detail the derivation of stochastic differential equations for the continuum time limit of the minority game. We show that all properties of the minority game can be understood by a careful theoretical analysis of such equations. In particular, (i) we confirm that the stationary state properties are given by the ground state configurations of a disordered (soft) spin system, (ii) we derive the full stationary state distribution, (iii) we characterize the dependence on initial conditions in the symmetric phase, and (iv) we clarify the behavior of the system as a function of the learning rate. This leaves us with a complete and coherent picture of the collective behavior of the minority game. Strikingly we find that the temperaturelike parameter, which is introduced in the choice behavior of *individual* agents turns out to play the role, at the *collective* level, of the inverse of a thermodynamic temperature.

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### I. INTRODUCTION

Even under the most demanding definition, the minority game (MG) [1,2] definitely qualifies as a complex system. The MG can be regarded as an Ising model for systems of heterogeneous adaptive agents, which interact via a global mechanism that entails competition for limited resource, as found, for instance, in biology and financial markets. In spite of more than three years of intense research, its rich dynamical behavior is still the subject of investigations; many variations of the basic MG are being proposed, each uncovering new surprising regions of phase space.

Most importantly, Refs. [3–7] have shown that much theoretical insight can be gained on the behavior of this class of models, using nonequilibrium statistical physics and statistical mechanics of disordered systems. The approach of Refs. [3–6] rests on the assumption that, in a continuum time limit (CTL), the dynamics of the MG can be described by a set of *deterministic* equations. From these, one derives a function  $H$  that is minimized along all trajectories; hence, the stationary state of the system corresponds to the ground state of  $H$ , which can be computed exactly by statistical mechanics techniques. This approach has been challenged in Refs. [8,9], which have proposed a *stochastic* dynamics for the MG, thus leading to some debate in the literature [10,11].

In this paper, our aim is to analyze in detail the derivation of the CTL in order to clarify this issue. We show that a proper derivation of the CTL indeed reconciles the two approaches: the resulting dynamical equations—Eqs. (15)–(17) below, which are our central result—are indeed *stochastic*, as suggested in Refs. [8,9], but still the stationary state of the dynamics is described by the minima of the function  $H$ , as suggested in Refs. [3,4]. We then confirm the analytic results derived previously. In few words, our analysis follows two main steps: first, we characterize the average behavior of agents by computing the frequency with which they play their strategies. This step can be translated in the study of the ground state properties of a soft spin disordered Hamiltonian.

Second, we characterize the fluctuations around the average behavior. To do this, we explicitly solve the Fokker-Planck equation associated with the stochastic dynamics.

The results we derive are the following.

(1) We derive the full probability distribution in the stationary state. Remarkably we find that the parameter that is introduced as a temperature in the individual choice model turns out to play the role of the inverse of a global temperature.

(2) For  $\alpha > \alpha_c$  the distribution factorizes over the agents whereas in the symmetric phase ( $\alpha < \alpha_c$ ) agents play in a correlated way. In the latter case, the correlations contribute to the stochastic force acting on agents. We show how the dependence of global efficiency on individual temperature found in Ref. [8] arises as a consequence of these correlations.

(3) We extend the analytic approach of Refs. [3,4] to the  $\alpha < \alpha_c$  phase and asymmetric initial conditions. The dependence on the initial conditions in this phase, first noticed and discussed in Refs. [3,4], has been more recently studied quantitatively in Refs. [9,7]. We clarify the origin of this behavior and derive analytic solutions in the limit  $\Gamma \rightarrow 0$ .

(4) We show that the stronger the initial asymmetry in agents' evaluation of their strategies, the larger is the efficiency and the more stable is the system against crowd effects [12].

(5) We derive the Hamiltonian of MGs with nonlinear payoffs.

This leaves us with a coherent picture of the collective behavior of the minority game, which is an important reference framework for the study of complex systems of heterogeneous adaptive agents.

### II. THE MODEL

The dynamics of the MG is defined in terms of dynamical variables  $U_{s,i}(t)$  in discrete time  $t=0,1,\dots$ . These are scores, propensities or “attractions” [14], which each agent

$i=1, \dots, N$  attaches to each of his possible choices  $s=1, \dots, S$ . Each agent takes a decision  $s_i(t)$  with

$$\text{Prob}\{s_i(t)=s\} = \frac{e^{\Gamma_i U_{s,i}(t)}}{\sum_{s'} e^{\Gamma_i U_{s',i}(t)}}, \quad (1)$$

where  $\Gamma_i > 0$  appears as an ‘‘individual inverse temperature.’’ The original MG corresponds to  $\Gamma_i = \infty$  [1] and was generalized later to  $\Gamma_i \equiv \Gamma < \infty$  [8].

The public information variable  $\mu(t)$  is given to all agents; it belongs to the set of integers  $(1, \dots, P)$  and can either be the binary encoding of the last  $M$  winning choices [1], or drawn at random from a uniform distribution [15]; we stick to the latter case for sake of simplicity.<sup>1</sup> The action  $a_{s_i(t),i}^{\mu(t)}$  of each agent depends on its choice  $s_i(t)$  and on  $\mu(t)$ . The coefficients  $a_{s,i}^{\mu}$ , called strategies, play the role of quenched disorder: they are randomly drawn signs ( $\text{Prob}\{a_{s,i}^{\mu} = \pm 1\} = 1/2$ ), independently for each  $i, s$ , and  $\mu$ . On the basis of the outcome

$$A(t) = \sum_{i=1}^N a_{s_i(t),i}^{\mu(t)}, \quad (2)$$

each agent updates his scores according to

$$U_{s,i}(t+1) = U_{s,i}(t) - a_{s,i}^{\mu(t)} \frac{A(t)}{P}. \quad (3)$$

The idea of this equation is that agents reward [ $U_{s,i}(t+1) > U_{s,i}(t)$ ] those strategies that would have predicted the *minority sign*— $A(t)/|A(t)|$ . The MG was initially proposed with a nonlinear dependence on  $A(t)$ , i.e., with a dynamics  $U_{s,i}(t+1) = U_{s,i}(t) - a_{s,i}^{\mu(t)} \text{sgn}[A(t)]$ . This leads to qualitatively similar results. The extension of our theory to nonlinear cases is dealt with in Appendix A. We shall not discuss any longer the interpretation of the model, which is discussed at length elsewhere [4,17–19].

The sources of randomness are in the choices of  $\mu(t)$  by nature and of  $s_i(t)$  by agents. These are fast fluctuating degrees of freedom. As a consequence also,  $U_{s,i}(t)$  and hence the probability with which agents chose  $s_i(t)$  are subject to stochastic fluctuations. Our analysis will indeed focus on the characterization of the low-frequency fluctuations of  $U_{s,i}$  by integrating out the high-frequency fluctuations of  $\mu(t)$  and  $s_i(t)$ . This will become clearer in the next section. For the time being let it suffice to say that there are two levels of fluctuations, that of ‘‘fast’’ variables  $\mu(t)$  and  $s_i(t)$  and that of ‘‘slow’’ degrees of freedom  $U_{s,i}(t)$ .

The key parameter is the ratio  $\alpha = P/N$  [18] and the two relevant quantities are

$$\sigma^2 = \langle A^2 \rangle, \quad H = \frac{1}{P} \sum_{\mu=1}^P \langle A | \mu \rangle^2, \quad (4)$$

<sup>1</sup>Both prescriptions lead to qualitatively similar results for the quantities we study here. See [16] for more details.

which measure, respectively, global efficiency and predictability.<sup>2</sup>

Generalizations of the model, where agents account for their market impact [3,4], where deterministic agents—so-called producers—are present [5], or where agents are allowed not to play [20–24], have been proposed. Rather than dealing with the most generic model, which would depend on too many parameters, we shall limit our discussion to the plain MG. Furthermore, we shall specialize, in the second part of the paper, to the case  $S=2$ , which lends itself to a simpler analytic treatment. The analysis carries through in obvious ways to the more general cases discussed in Refs. [4,5,3,24].

### III. THE CONTINUUM TIME LIMIT

Our approach, which follows that of Refs. [3,4], is based on two key observations.

(1) The scaling  $\sigma^2 \sim N$ , at fixed  $\alpha$ , suggests that typically  $A(t) \sim \sqrt{N}$ . Hence time increments of  $U_{s,i}(t)$  in Eq. (3) are small (i.e., of order  $\sqrt{N}/P \sim 1/\sqrt{N}$ ).

(2) Characteristic times of the dynamics are proportional to  $P$ . Naively this is because agents need to ‘‘test’’ their strategies against all  $P$  values of  $\mu$ , which requires of the order of  $P$  time steps. More precisely, one can reach this conclusion by measuring relaxation or correlation times and verifying that they indeed grow linearly with  $P$  (see Ref. [10]).

The second observation implies that one needs to study the dynamics in the rescaled time  $\tau = t/P$ . This makes our approach differ from that of Refs. [8,9], where the time is not rescaled.

In order to study the dynamics for  $P, N \gg 1$  at fixed  $\alpha$ , we shall focus on a fixed small increment  $d\tau$  such that  $Pd\tau = \alpha Nd\tau \gg 1$ . This means that we take the continuum time limit  $d\tau \rightarrow 0$  *only after* the thermodynamic limit  $N \rightarrow \infty$ . We focus only on the leading order in  $N$ . Furthermore, we shall also consider  $\Gamma_i$  finite and

$$\Gamma_i d\tau \ll 1, \quad (5)$$

which means that the limit  $\Gamma_i \rightarrow \infty$  should be taken after the limit  $d\tau \rightarrow 0$ . The orders in which these limits are taken, given the agreement with numerical simulation results, does not really matter: as we shall see differences only enter in the finite size corrections. We shall come back later to these issues.

Iteration of the dynamics for  $Pd\tau$  time steps, from  $t = P\tau$  to  $t = P(\tau + d\tau)$  gives

$$u_{s,i}(\tau + d\tau) - u_{s,i}(\tau) = -\frac{1}{P} \sum_{t=P\tau}^{P(\tau+d\tau)-1} a_{s,i}^{\mu(t)} A(t), \quad (6)$$

where we have introduced the functions  $u_{s,i}(\tau) = U_{s,i}(P\tau)$ .

<sup>2</sup>Averages  $\langle \dots \rangle$  stand for time averages in the stationary state of the process. Then  $\langle \dots | \mu \rangle$  stands for time averages conditional on  $\mu(t) = \mu$ .

Let us separate a deterministic ( $du_{s,i}$ ) from a stochastic ( $dW_{s,i}$ ) term in this equation by replacing

$$a_{s,i}^{\mu(t)} A(t) = \overline{a_{s,i} \langle A \rangle_{\pi}} + X_{s,i}(t). \quad (7)$$

Here and henceforth, we denote averages over  $\mu$  by an overline

$$\overline{R} = \frac{1}{P} \sum_{\mu=1}^P R^{\mu},$$

while  $\langle \dots \rangle_{\pi}$  stands for an average over the distributions

$$\pi_{s,i}(\tau) = \frac{1}{Pd\tau} \sum_{t=P\tau}^{P(\tau+d\tau)-1} \frac{e^{\Gamma_i U_{s,i}(t)}}{\sum_r e^{\Gamma_i U_{r,i}(t)}}, \quad (8)$$

which is the frequency with which agent  $i$  plays strategy  $s$  in the time interval  $P\tau \leq t < P(\tau + d\tau)$ .

Notice that  $\pi_{s,i}(\tau)$  will themselves be stochastic variables, hence we also define the average on the stationary state as

$$\langle \dots \rangle = \lim_{\tau_0, T \rightarrow \infty} \frac{1}{T} \int_{\tau_0}^{\tau_0+T} d\tau \langle \dots \rangle_{\pi(\tau)}, \quad (9)$$

where the average inside the integral is performed with the probabilities  $\pi_{s,i}(\tau)$ .

Hence

$$\begin{aligned} u_{s,i}(\tau + d\tau) - u_{s,i}(\tau) &= du_{s,i}(\tau) + dW_{s,i}(\tau) \\ &= -\overline{a_{s,i} \langle A \rangle_{\pi}} d\tau \\ &\quad + \frac{1}{P} \sum_{t=P\tau}^{P(\tau+d\tau)-1} X_{s,i}(t). \end{aligned} \quad (10)$$

Now the first term is of order  $d\tau$  as required for a deterministic term. In addition it remains finite as  $N \rightarrow \infty$  [25].

The second term is a sum of  $Pd\tau$  random variables  $X_{s,i}(t)$  with zero average. We take  $d\tau$  fixed and  $N$  very large, so that  $Pd\tau \gg 1$  and we can use limit theorems. The variables  $X_{s,i}(t)$  are independent from time to time, because both  $\mu(t)$  and  $s_j(t)$  are drawn independently at each time. Hence  $X_{s,i}(t)$  for  $P\tau \leq t < P(\tau + d\tau)$  are independent and identically<sup>3</sup> distributed. For  $Pd\tau \gg 1$  we may approximate the second term  $dW_{s,i}$  of Eq. (10) by a Gaussian variable with zero average and variance,

$$\begin{aligned} \langle dW_{s,i}(\tau) dW_{r,j}(\tau') \rangle &= \frac{\delta(\tau - \tau')}{P^2} \sum_{t=P\tau}^{P(\tau+d\tau)} \langle X_{s,i}(t) X_{r,j}(t) \rangle_{\pi} \\ &= \delta(\tau - \tau') d\tau \frac{\langle X_{s,i}(t) X_{r,j}(t) \rangle_{\pi}}{P}, \end{aligned}$$

<sup>3</sup>Strictly speaking,  $s_i(t)$  is drawn from the distribution in Eq. (1) and not from  $\pi_{s,i}$  of Eq. (8). However, these two distributions differ by a negligible amount as long as the condition (5) holds (see later).

where the  $\delta(\tau - \tau')$  comes from independence of  $X_{s,i}(t)$  and  $X_{r,j}(t')$  for  $t \neq t'$  and the fact that  $X_{s,i}(t)$  are identically distributed in time leads to the expression in the second line. Now

$$\frac{\langle X_{s,i}(t) X_{r,j}(t) \rangle_{\pi}}{P} = \frac{\overline{a_{s,i} a_{r,j} \langle A^2 \rangle_{\pi}}}{P} - \frac{\overline{a_{s,i} \langle A \rangle_{\pi}} \overline{a_{r,j} \langle A \rangle_{\pi}}}{P}. \quad (11)$$

The second term always vanishes for  $N \rightarrow \infty$  because  $\overline{a_{s,i} \langle A \rangle_{\pi}}$  is of order  $N^0$  [25]. In the first term, instead,

$$\langle A^2 | \mu \rangle_{\pi} = N + \sum_{k \neq l=1}^N \sum_{s',r'} a_{s',k}^{\mu} a_{r',l}^{\mu} \pi_{s',k} \pi_{r',l} \quad (12)$$

is of order  $N$ , which then gives a positive contribution in Eq. (11) for  $N \rightarrow \infty$ .

Equation (12) leads to a stochastic dynamics where the noise covariance depends on the stochastic variables  $u_{s,i}(\tau)$  themselves. The complications that result from this fact can be avoided if one takes the approximation

$$\langle A^2 | \mu \rangle_{\pi} \approx \overline{\langle A^2 \rangle} \equiv \sigma^2. \quad (13)$$

This approximation can be justified naively by observing that the dependence on  $\pi_{s,i}$  of the correlations only involves the global quantity  $\langle A^2 | \mu \rangle_{\pi} / P$  for which one expects some sort of self-averaging properties. Numerical results suggest that terms that are ignored by Eq. (13) are negligible for  $N \gg 1$ , but we were unable to prove this in general [26].

Within this approximation the correlation, for  $N \gg 1$  becomes

$$\langle dW_{s,i}(\tau) dW_{r,j}(\tau') \rangle \cong \frac{\sigma^2}{\alpha N} \overline{a_{s,i} a_{r,j}} \delta(\tau - \tau') d\tau. \quad (14)$$

Note that, for  $r \neq s$  or  $j \neq i$ , correlations  $\langle dW_{s,i}(\tau) dW_{r,j}(\tau') \rangle \propto \overline{a_{s,i} a_{r,j}} \sim 1/\sqrt{N}$  vanish as  $N \rightarrow \infty$ . However, it is important to keep the off-diagonal terms because they keep the dynamics of the phase space point  $|U(t)\rangle = \{U_{s,i}(t)\}_{s=1, \dots, S, i=1, \dots, N}$  constrained to the linear space spanned by the vectors  $|a^{\mu}\rangle = \{a_{s,i}^{\mu}\}_{s=1, \dots, S, i=1, \dots, N}$ , which contains the initial condition  $|U(0)\rangle$ . The original dynamics of  $U_{s,i}(t)$  indeed possesses this property.

It is important to remark that the approximation (13) makes our approach a self-consistent theory for  $\sigma^2$ . We introduce  $\sigma^2$  as a constant in Eq. (13), which then has to be computed self-consistently from the dynamic equations.

Summarizing, the dynamics of  $u_{s,i}$  is described by a continuum time Langevin equation,

$$\frac{du_{s,i}(\tau)}{d\tau} = -\overline{a_{s,i} \langle A \rangle} + \eta_{s,i}(\tau), \quad (15)$$

$$\langle \eta_{s,i}(\tau) \rangle = 0, \quad (16)$$

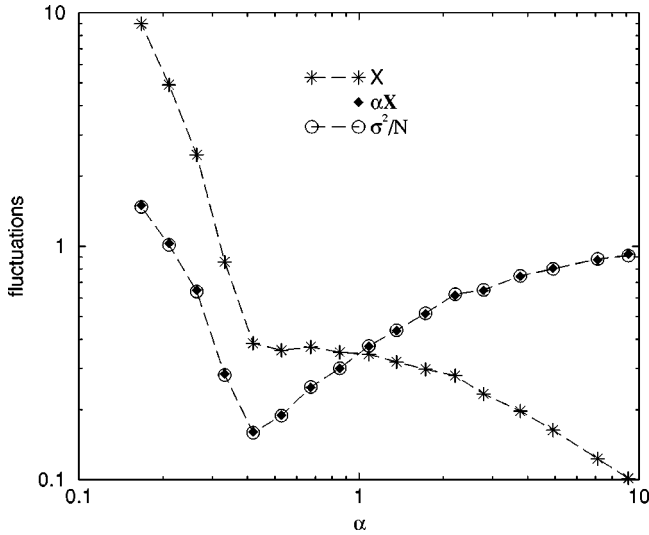


FIG. 1. Noise strength averaged over all agents (stars); when it is multiplied by  $\alpha$  (diamonds), one recovers  $\sigma^2/N$  (circles) ( $P = 32$ ,  $S = 2$ ,  $300P$  iterations,  $\Gamma = \infty$ , average over 50 samples). Dashed lines are for eye guidance only.

$$\langle \eta_{s,i}(\tau) \eta_{r,j}(\tau') \rangle \cong \frac{\sigma^2}{\alpha N} a_{s,i} a_{r,j} \delta(\tau - \tau'). \quad (17)$$

Equation (15), given its derivation, has to be interpreted in the Ito sense. The expression for the noise strength is confirmed by Fig. 1, where the measure of  $X = \sum_{i,s} \langle X_{i,s}^2 \rangle / (PNS)$  in a MG is reported; note that these numerical simulations were done for  $\Gamma = \infty$  and confirm Eq. (17) is valid even for  $\alpha < \alpha_c$ . Figure 2 compares the results of numerical simulations of the MG, as a function of  $\Gamma$ , with those of a semianalytic solution of Eqs. (15)–(17), to be discussed later. The agreement of the two approaches shows that Eqs. (15)–(17) are valid even in the symmetric phase ( $\alpha \leq \alpha_c$ ) for all values of  $\Gamma$ .

The instantaneous probability distribution of  $s$ , in the continuum time limit, reads

$$\pi_{s,i}(\tau) = \frac{e^{\Gamma_i u_{s,i}(\tau)}}{\sum_r e^{\Gamma_i u_{r,i}(\tau)}}. \quad (18)$$

This and Ito calculus then lead to a dynamic equation for  $\pi_{s,i}(\tau)$ . We prefer to exhibit this for  $\Gamma_i = \Gamma$  and using the rescaled time  $t = \Gamma\tau$ ,

$$\begin{aligned} \frac{d\pi_{s,i}}{dt} = & -\pi_{s,i} [\overline{a_{s,i} \langle A \rangle} - \vec{\pi}_i \cdot \vec{a}_i \langle A \rangle] + \frac{\sigma^2 \Gamma}{\alpha N} \pi_{s,i} (\pi_{s,i} - \vec{\pi}_i^2) \\ & + \sqrt{\Gamma} \pi_{s,i} (\eta_{s,i} - \vec{\pi}_i \vec{\eta}_i). \end{aligned} \quad (19)$$

The first term in the right-hand side comes from the deterministic part of Eq. (15), the second from the Ito term (where we neglected terms proportional to  $a_{i,s} a_{i,s'} \sim 1/\sqrt{N}$  for  $s \neq s'$ ), and the third from the stochastic part. It is clear that, in the limit  $\Gamma \rightarrow 0$  the last two terms vanish and the dynamics becomes deterministic.

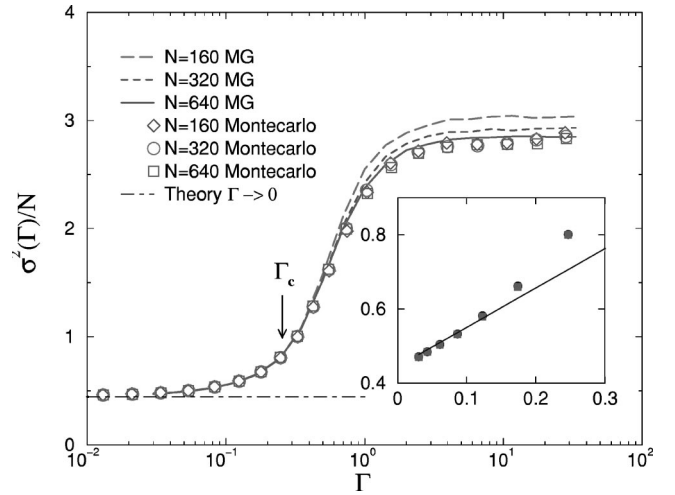


FIG. 2. Global efficiency  $\sigma^2/N$  versus  $\Gamma$  for  $\alpha = 0.1 < \alpha_c$  and different system sizes. Lines refer to direct simulations of the MG with  $N = 160, 320$ , and  $640$ . Finite size effect for  $\Gamma \gg 1$  are evident. Symbols refer instead to the solution of the self-consistent equation (36) for the same system sizes. For both methods and all values of  $N$ ,  $\sigma^2$  is averaged over 100 realizations of the disorder. The arrow marks the location of  $\Gamma_c$  predicted by Eq. (34). In the inset, the theoretical prediction, Eq. (37), on the leading behavior of  $\sigma^2/N$  for  $\Gamma \ll 1$  (solid line) is tested against numerical simulations of the MG (points) for the same values of  $N$ .

We see then that  $\Gamma$  tunes the strength of stochastic fluctuations in much the same way as temperature does for thermal fluctuations in statistical mechanics. The “individual inverse temperature”  $\Gamma$  should indeed more correctly be interpreted as a learning rate.<sup>4</sup> Furthermore, it plays the role of a “global temperature.” We shall pursue this discussion in detail below for the case  $S = 2$ .

At this point, let us comment on the limit  $\Gamma \rightarrow \infty$ , which is of particular importance since it corresponds to the original MG. It is clear that in the limit  $\Gamma \rightarrow \infty$  the dynamical Eqs. (19) become problematic. The origin of the problem lies in the order in which the limits  $N \rightarrow \infty$  and  $\Gamma \rightarrow \infty$  is performed. Indeed in Eq. (8)  $U_{s,i}(t) \approx u_{s,i}(\tau) + O(d\tau)$  for  $P\tau \leq t < P(\tau + d\tau)$ . Therefore, as long as  $\Gamma d\tau \ll 1$  the difference between Eq. (8) and  $\pi_{s,i}$  in Eq. (18) is negligible. In practice, in order to satisfy both  $\Gamma d\tau \ll 1$  and  $Pd\tau \gg 1$ , one needs  $\Gamma \ll P$ . When this condition is not satisfied the instantaneous probability (1) fluctuates very rapidly at each time step. Equation (8) averages out these high-frequency fluctuations so that, even for  $\Gamma = \infty$ , the distribution  $\pi_{s,i}(\tau)$  of Eq. (8) is not a discontinuous step function of  $u_{s,i}(\tau)$ , as suggested by Eq. (18). High-frequency fluctuations contribute to the functional form of  $\pi_{s,i}$  on  $u_{s,i}$ , which will differ from Eq. (18).

Summarizing, when we let  $\Gamma \rightarrow \infty$  only after the limit  $N \rightarrow \infty$  has been taken, no problem arises. There is no reason to

<sup>4</sup>This is an *a posteriori* learning rate. Indeed  $1/\Gamma$  is the time the dynamics of the scores needs in order to learn a payoff difference. From a different viewpoint,  $\Gamma$  tunes the randomness of the response of agents. The larger the randomness, the longer it takes to average fluctuations out.



believe that results change if the order of the limits is interchanged. This expectation, as we shall see, is confirmed by numerical simulations (see Fig. 2): direct numerical simulations of the MG deviate from the prediction of Eqs. (15)–(17) only for finite size effects that vanish as  $N \rightarrow \infty$ .

Equations (15)–(17) are our central result. We shall devote the rest of the paper to discuss their content and to show that all of the observed behavior of the MG can be derived from these equations.

#### IV. STATIONARY STATE

Let us take the average, denoted by  $\langle \cdot \cdot \cdot \rangle$ , of Eq. (15) on the stationary state (SS). Let

$$f_{s,i} = \langle \pi_{s,i} \rangle$$

be the frequency with which agent  $i$  plays strategy  $s$  in the SS. Then we have

$$v_{s,i} \equiv \left\langle \frac{du_{s,i}}{d\tau} \right\rangle = -\overline{a_{s,i} \langle A \rangle}, \quad \langle A | \mu \rangle = \sum_{j,s'} f_{s',j} a_{s',j}^\mu.$$

Given the relation between  $\pi_{s,i}$  and  $u_{s,i}$  and considering that the long-time dynamics of  $u_{s,i}$  in the SS is  $u_{s,i}(\tau) = \text{const} + v_{s,i}\tau$ , we have that (i) each strategy that is played in the SS by agent  $i$  must have the same “velocity”  $v_{s,i} = v_i^*$  and (ii) strategies that are not played (i.e., with  $f_{s,i} = 0$ ) must have  $v_{s,i} < v_i^*$ . In other words

$$-\overline{a_{s,i} \langle A \rangle} = v_i^* \quad \forall i,s \quad \text{such that } f_{s,i} > 0, \quad (20)$$

$$-\overline{a_{s,i} \langle A \rangle} \leq v_i^* \quad \forall i,s \quad \text{such that } f_{s,i} = 0. \quad (21)$$

Consider now the problem of constrained minimization of  $H$  in Eq. (4), subject to  $f_{s,i} \geq 0$  for all  $s,i$  and the normalization conditions. Introducing Lagrange multipliers  $\lambda_i$  to enforce  $\sum_s f_{s,i} = 1$  for all  $i$ , this problem reads

$$\min_{\{f_{s,i} \geq 0\}} \left\{ \overline{\langle A \rangle^2} - \sum_{i=1}^N \lambda_i \left( 1 - \sum_{s=1}^S f_{s,i} \right) \right\}. \quad (22)$$

Taking derivatives, we find that if  $f_{s,i} > 0$  then  $\overline{a_{s,i} \langle A \rangle} + \lambda_i = 0$  whereas if  $f_{s,i} = 0$  then  $\overline{a_{s,i} \langle A \rangle} + \lambda_i \geq 0$ . These are exactly Eqs. (20) and (21) where  $v_i^* = \lambda_i$ . We then conclude that the two problems, Eqs. (20) and (21) and Eq. (22) are one and the same problem.<sup>5</sup> In other words  $f_{s,i}$  can be computed from the constrained minimization of  $H$  as proposed in Refs. [3,4].

<sup>5</sup>Indeed both problems can be put in the form of a linear complementarity problem [27],

$$\begin{aligned} f_{s,i} &\geq 0, \\ \sum_{j,s'} \overline{a_{s,i} a_{s',j} f_{s',j}} + v_i^* &\geq 0, \\ f_{s,i} \left[ \sum_{j,s'} \overline{a_{s,i} a_{s',j} f_{s',j}} + v_i^* \right] &= 0. \end{aligned}$$

This problem has a solution for all values of  $v_i^*$  because of non-negativity of the matrix  $\overline{a_{s,i} a_{s',j}}$ , see Ref. [27].

Hence the statistical mechanics approach based on the study of the ground state of  $H$  is correct. This approach gives the frequency  $f_{s,i}$  with which agents play their strategies.

We remark once more that  $H$  is a function of the stationary state probabilities  $f_{s,i}$ . Also note that

$$\tilde{H}\{\pi_{s,i}\} = \sum_{i,j=1}^N \sum_{s,s'=1}^S \overline{a_{s,i} a_{s',j}} \pi_{s,i} \pi_{s',j}$$

as a function of the instantaneous probabilities  $\pi_{s,i}$  is *not* a Lyapunov function of the dynamics. The dynamical variables  $\pi_{s,i}(t)$  are subject to stochastic fluctuations of the order of  $\sqrt{\Gamma_i}$  around their average values  $f_{s,i}$ . Only in the limit  $\Gamma_i \rightarrow 0$ , when the dynamics becomes deterministic and  $\pi_{s,i} \rightarrow f_{s,i}$ , the quantity  $\tilde{H}\{\pi_{s,i}\}$  becomes a Lyapunov function.

The solution to the minimization of  $H$  reveals two qualitatively distinct phases [3,4] that are separated by a phase transition occurring as  $\alpha \rightarrow \alpha_c$ . We discuss qualitatively the behavior of the solution for a generic  $S$  and leave for the next section a more detailed discussion in the simpler case  $S=2$ .

#### A. Independence on $\Gamma$ for $\alpha > \alpha_c$

When  $\alpha > \alpha_c$  the solution to Eq. (22) is unique and  $H > 0$ . Hence  $f_{s,i}$  does not depend on  $\Gamma$ , neither does  $H$ . In addition we shall see that

$$\langle \pi_{s,i} \pi_{s',j} \rangle = \langle \pi_{s,i} \rangle \langle \pi_{s',j} \rangle = f_{s,i} f_{s',j} \quad \text{for } i \neq j, \quad (23)$$

implying that

$$\sigma^2 \equiv N + \sum_{i \neq j} \sum_{s,r} \overline{a_{s,i} a_{r,j}} \langle \pi_{s,i} \pi_{r,j} \rangle$$

does not depend on  $\Gamma$  either. Hence the solution  $\{f_{s,i}\}$  uniquely determines all quantities in the SS, as well as the parameters that enter into the dynamics [notice the dependence on  $\sigma^2$  in Eq. (17)]. In particular,  $\sigma^2$  does not depend on  $\Gamma$ .

#### B. Dependence on $\Gamma$ and on initial conditions for $\alpha < \alpha_c$

For  $\alpha < \alpha_c$  the solution to the minimization problem is not unique: there is a connected set of points  $\{f_{s,i}\}$  such that  $H = 0$ . Let us first discuss the behavior of the system in the limit  $\Gamma \rightarrow 0$ , where the dynamics becomes deterministic. The dynamics reaches a stationary state  $\{f_{s,i}\}$  that depends on the initial conditions.

In order to see this, let us introduce the vector notation  $|v\rangle = \{v_{s,i}, s=1, \dots, S, i=1, \dots, N\}$ . Then for all times  $|u(\tau)\rangle$  is of the form

$$|u(\tau)\rangle = |u(0)\rangle + \sum_{\mu=1}^P |a^\mu\rangle C^\mu(\tau),$$

where  $C^\mu(\tau)$  are  $P$  functions of time.

If there are vectors  $\langle v|$  such that  $\langle v|a^\mu\rangle = 0$  for all  $\mu$ , then  $\langle v|u(\tau)\rangle = \langle v|u(0)\rangle$ , i.e., the components of the scores will not change at all along these vectors. As a result the SS will

depend on initial conditions  $|u(0)\rangle$ . These vectors  $\langle v|$  exist exactly for  $\alpha < \alpha_c$  [4], because the ‘‘dimensionality’’ of the vectors  $|u(\tau)\rangle$  is larger than  $P$ .<sup>6</sup>

The picture is made even more complex by the fact that for  $\alpha < \alpha_c$ , when  $\Gamma$  is finite, Eq. (23) does not hold. Hence  $\sigma^2$  has a contribution, which depends on the stochastic fluctuations around  $f_{s,i}$ . The strength of these fluctuations, given by Eqs. (17) and (19), depends on  $\Gamma$  and  $\sigma^2$  itself. We face, in this case, a self-consistent problem:  $\sigma^2$  enters as a parameter of the dynamics but should be computed in the stationary state of the dynamics itself. Therefore, the solution to this problem and hence  $\sigma^2$  depends on  $\Gamma$ . The solution  $\{f_{s,i}\}$  to the minimization of  $H$  should also be computed self-consistently. As a result, the SS properties acquire a dependence on  $\Gamma$ .

The condition (23), which is similar to the *clustering* property in spin glasses [28], plays then a crucial role. We show below how the condition (23), the dependence on initial conditions, and on  $\Gamma$  enter into the detailed solution for  $S=2$ . By similar arguments our conclusion can be generalized to all  $S > 2$ .

## V. THE CASE $S=2$

We work in this section with the simpler case of  $S=2$  strategies, labeled by  $s = \pm$ . We also set  $\Gamma_i = \Gamma$  for all  $i$ . Following Refs. [3,13] we introduce the variables

$$\xi_i^\mu = \frac{a_{+,i}^\mu - a_{-,i}^\mu}{2}, \quad \Omega^\mu = \sum_{i=1}^N \frac{a_{+,i}^\mu + a_{-,i}^\mu}{2}.$$

Let us rescale time  $t = \Gamma \tau$  and introduce the variables

$$y_i(t) = \Gamma \frac{u_{+,i}(\tau) - u_{-,i}(\tau)}{2}.$$

Then, using Eq. (8), the dynamical equations (15)–(17) become

$$\frac{dy_i}{dt} = -\overline{\xi_i \Omega} - \sum_{j=1}^N \overline{\xi_i \xi_j \tanh(y_j)} + \zeta_i, \quad (24)$$

$$\langle \zeta_i(t) \zeta_j(t') \rangle = \frac{\Gamma \sigma^2}{\alpha N} \overline{\xi_i \xi_j} \delta(t - t'). \quad (25)$$

The Fokker-Planck (FP) equation for the probability distribution  $P(\{y_i\}, t)$  under this dynamics reads

<sup>6</sup>In order to compute the dimensionality of the vectors  $|u\rangle$  we have to take into account the  $N$  normalization conditions and the fact that strategies that are not played ( $f_{s,i} = 0$ ) should not be counted. So if there are  $N_>$  variables  $f_{s,i} > 0$ , the relevant dimension of the space of  $|u\rangle$  is  $N_> - N$ . Hence vectors  $\langle v|$  orthogonal to all  $|a^\mu\rangle$  exist for  $N_> - N > P$ , i.e., for  $\alpha < \alpha_c = N_>(\alpha_c)/N - 1$ .

$$\begin{aligned} \frac{\partial P(\{y_i\}, t)}{\partial t} = & \sum_{i=1}^N \frac{\partial}{\partial y_i} \left\{ \overline{\xi_i \Omega} + \sum_{j=1}^N \overline{\xi_i \xi_j \tanh(y_j)} \right. \\ & \left. + \frac{1}{\beta} \sum_{j=1}^N \overline{\xi_i \xi_j} \frac{\partial}{\partial y_j} \right\} P(\{y_i\}, t), \end{aligned} \quad (26)$$

where we have introduced the parameter

$$\beta = \frac{2\alpha N}{\Gamma \sigma^2}. \quad (27)$$

Multiply Eq. (26) by  $y_i$  and integrate over all variables. Using integration by parts, assuming that  $P \rightarrow 0$  fast as  $y_j \rightarrow \infty$ , one gets

$$\frac{\partial \langle y_i \rangle}{\partial t} = -\overline{\xi_i \Omega} - \sum_{j=1}^N \overline{\xi_i \xi_j} \langle \tanh(y_j) \rangle.$$

Let us look for solutions with  $\langle y_i \rangle \sim v_i t$  and define  $m_i = \langle \tanh(y_i) \rangle$ . Hence for  $t \rightarrow \infty$  we have

$$v_i = -\overline{\xi_i \Omega} - \sum_{j=1}^N \overline{\xi_i \xi_j} m_j. \quad (28)$$

Now, either  $v_i = 0$  and  $\langle y_i \rangle$  is finite or  $v_i \neq 0$ , which means that  $y_i \rightarrow \pm \infty$  and  $m_i = \text{sgn } v_i$ . In the latter case ( $v_i \neq 0$ ) we say that agent  $i$  is *frozen* [13], we call  $\mathcal{F}$  the set of frozen agents and  $\phi = |\mathcal{F}|/N$  the fraction of frozen agents.

As in the general case, the parameters  $v_i$  for  $i \in \mathcal{F}$  and  $m_i \equiv \langle \tanh(y_i) \rangle$  for  $i \notin \mathcal{F}$  are obtained by solving the constrained minimization of

$$H = \frac{1}{P} \sum_{\mu=1}^P \left[ \Omega^\mu + \sum_{i=1}^N \xi_i^\mu m_i \right]^2.$$

When the solution of  $\min H$  is unique, i.e., for  $\alpha > \alpha_c$ , the parameters  $m_i$  depend only on the realization of disorder  $\{\xi_i^\mu, \Omega^\mu\}$ , and their distribution can be computed as in Ref. [3]. When the solution is not unique, i.e., for  $\alpha < \alpha_c$ , we are left with the problem of finding which solution the dynamics selects. Let us suppose that we have solved this problem (we shall come back later to this issue) so that all  $m_i$  are known.

Using the stationary condition Eq. (28), we can write the FP equation for the probability distribution  $P_u(y_i, i \notin \mathcal{F})$  of unfrozen agents. For times so large that all agents in  $\mathcal{F}$  are indeed frozen [i.e.,  $s_i(t) = \text{sgn } v_i$ ] this reads

$$\frac{\partial P_u}{\partial t} = \sum_{i \notin \mathcal{F}} \frac{\partial}{\partial y_i} \sum_{j \notin \mathcal{F}} \overline{\xi_i \xi_j} \left\{ \tanh(y_j) - m_j + \frac{1}{\beta} \frac{\partial}{\partial y_j} \right\} P_u.$$

This has a solution

$$P_u \propto \exp \left\{ -\beta \sum_{j \notin \mathcal{F}} [\ln \cosh y_j - m_j y_j] \right\}. \quad (29)$$

Finally we have to impose the constraint that  $|y(t)\rangle = \{y_i(t)\}_{i=1}^N$  must lie on the linear space spanned by the vectors  $|\xi^\mu\rangle$ , which contains the initial condition  $|y(0)\rangle$ . This means that

$$P_u \propto \mathcal{P}_{y(0)} \exp\left\{-\beta \sum_{j \notin \mathcal{F}} [\ln \cosh y_j - m_j y_j]\right\}, \quad (30)$$

where the projector  $\mathcal{P}_{y(0)}$  is given by

$$\mathcal{P}_{y(0)} \equiv \prod_{\mu=1}^P \int_{-\infty}^{\infty} dc^\mu \prod_{i=1}^N \delta\left[y_i - y_i(0) - \sum_{\mu=1}^P c^\mu \xi_i^\mu\right]. \quad (31)$$

We find it remarkable that  $\Gamma$ , which is introduced as the inverse of an *individual* ‘‘temperature’’ in the definition of the model, actually turns out to be proportional to  $\beta^{-1}$  [see Eq. (27)], which plays *collectively* a role quite similar to that of temperature.

Using the distribution Eq. (30), we can compute

$$\begin{aligned} \sigma^2 = H + \sum_{i=1}^N \overline{\xi_i^2} (1 - m_i^2) + \sum_{i \neq j} \overline{\xi_i \xi_j} \langle (\tanh y_i - m_i) \\ \times (\tanh y_j - m_j) \rangle. \end{aligned} \quad (32)$$

This depends on  $\beta$ , i.e., on  $\sigma^2$  itself by virtue of Eq. (27). The stationary state is then the solution of a self-consistent problem. Let us analyze in detail the solution of this self-consistent problem.

### A. $\alpha > \alpha_c$

For  $\alpha > \alpha_c$  the solution of  $\min H$  is unique, and hence  $m_i$  depends only on the realization of the disorder. In addition, the number  $N - |\mathcal{F}| \equiv N(1 - \phi)$  of unfrozen agents is less than  $P$  and the constraint is ineffective, i.e.,  $\mathcal{P}_{y(0)} \equiv 1$ . The scores  $|y\rangle$  of unfrozen agents span a linear space, which is embedded in the one spanned by the vectors  $|\xi^\mu\rangle$ . Hence the dependence on initial conditions  $y_i(0)$  drops out. Therefore, the probability distribution of  $y_i$  factorizes as in Eq. (29). Then the third term of Eq. (32), which is the only one that depends on  $\beta$ , vanishes identically. We conclude that, for  $\alpha > \alpha_c$ ,  $\sigma^2$  only depends on  $m_i$  and is hence independent of  $\Gamma$  as confirmed by numerical simulations.

Summarizing, for  $\alpha > \alpha_c$  one derives a complete solution of the MG by finding first the minimum  $\{m_i\}$  of  $H$  and then by computing  $\sigma^2$ ,  $\beta$ , and the full distribution of  $y_i$  from Eq. (29).

### B. $\alpha \leq \alpha_c$ : Dependence on $\Gamma$ and crowd effects

When  $\alpha < 1 - \phi$ , on the other hand, the solution of  $\min H$  is not unique. Furthermore, the constraint cannot be integrated out and the stationary distribution depends on the initial conditions.

Numerical simulations [8] show that  $\sigma^2$  increases with  $\Gamma$  for  $\alpha < \alpha_c$  (see Fig. 2). This effect has been related to crowd effects in financial markets [12]. References [6] has shown that crowd effects can be fully understood in the limit  $\alpha$

$\rightarrow 0$ : as  $\Gamma$  exceeds a critical learning rate  $\Gamma_c$ , the time-independent SS becomes unstable and a *bifurcation* to a period-2 orbit occurs. Neglecting the stochastic term  $\zeta_i$ , Ref. [6] shows that this picture can be extended to  $\alpha > 0$ .<sup>7</sup> This approach suggests a crossover to a ‘‘turbulent’’ dynamics for  $\Gamma > \Gamma_c$ , where

$$\Gamma_c(\alpha) = \frac{4}{(1 + \sqrt{\alpha})^2 (1 - Q)} \quad (34)$$

and

$$Q = \frac{1}{N} \sum_{i=1}^N m_i^2.$$

Both  $Q$  and  $\Gamma_c$  can be computed exactly in the limit  $N \rightarrow \infty$  within the statistical mechanics approach [3,6].

This approach however (i) does not properly takes into account the stochastic term, (ii) does not explain what happens for  $\Gamma > \Gamma_c$ , and (iii) does not explain why such effects occur only for  $\alpha < \alpha_c$ .

The stochastic dynamics derived previously gives detailed answers to all these issues. We first restrict attention to symmetric initial conditions  $y_i(0) = 0$  and then discuss the dependence on initial conditions. The choice of  $y_i(0) = 0 \forall i$  is convenient because it allows one to use this same symmetry to identify the solution  $\{m_i\}$  to the minimization of  $H$ , independently of  $\Gamma$ . To be more precise, one can introduce a ‘‘magnetization’’

$$M \equiv \lim_{t \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \langle s_i(t) s_i(0) \rangle = \frac{1}{N} \sum_{i=1}^N m_i, \quad (35)$$

which measures the overlap of the SS configuration with the initial condition. Symmetric initial conditions are related to  $M = 0$  SS. These are the states we focus on. The solution is derived in two steps: (1) find the minimum  $\{m_i\}$  of  $H$  with  $M = 0$ , (2) compute self-consistently  $\sigma^2$ . The numerical pro-

<sup>7</sup>The idea of Ref. [6] is the following: imagine that our system is close to a SS point  $y_i^*$  at time  $t = t_k$ , when  $\mu(t) = 1$ . Will the system be close to  $y_i^*$  when the pattern  $\mu = 1$  occurs again the next time  $t' = t_{k+1}$ ? To see this, let us integrate Eq. (24) from  $t = t_k$  to  $t_{k+1}$ . In doing this we (i) neglect the noise term (i.e.,  $\zeta_i = 0$ ) and (ii) assume that  $\tanh y_i(t) \approx \tanh y_i(t_k)$  stays constant in the integration time interval. This latter assumption is similar to the recently introduced [7] batch version of the MG, where agents update their strategies every  $P$  time steps. This leads to the study of a discrete time dynamical system

$$y_i(t_{k+1}) = y_i(t_k) - \Gamma \left\{ \overline{\Omega \xi_i} + \sum_{j=1}^N \overline{\xi_i \xi_j} \tanh[y_j(t_k)] \right\}, \quad (33)$$

where the factor  $\Gamma$  comes because  $t_{k+1} - t_k$  is on average equal to  $\Gamma$ . The linear stability of fixed point solutions is analyzed setting  $y_i(t_k) = y_i^* + \delta y_i(k)$  and computing the eigenvalues of the linearized map  $\delta y_i(k+1) \approx \sum_j T_{i,j} \delta y_j(k)$ . There is a critical value of  $\Gamma$  above which the solution  $y_i^*$  become unstable, which is given by Eq. (34).

cedure for solving the problem is the following: given the realization of disorder  $\{\xi_i^\mu, \Omega^\mu\}$ , step (1)—finding the minimum  $\{m_i\}$  of  $H$ —is straightforward. For step (2) we sample the distribution (30) with the Monte Carlo method<sup>8</sup> at inverse temperature  $\beta$  and measure the  $\beta$ -dependent contribution of  $\sigma^2$  in Eq. (32),

$$\Sigma(\beta) = \sum_{i \neq j} \overline{\xi_i \xi_j} \langle (\tanh y_i - m_i)(\tanh y_j - m_j) \rangle_\beta.$$

Here  $\langle \dots \rangle_\beta$  stands for an average over the distribution (30) with parameter  $\beta$ . Finally we solve the equation

$$\sigma^2(\Gamma) = \sigma^2(0) + \Sigma \left[ \frac{2\alpha N}{\Gamma \sigma^2(\Gamma)} \right]. \quad (36)$$

This procedure was carried out for different system sizes and several values of  $\Gamma$ . The results, shown in Fig. 2, agree perfectly well with direct numerical simulations of the MG. Actually Fig. 2 shows that, for  $\Gamma \gg 1$ , the solution of the self-consistent equation (36) suffers much lesser finite size effects than the direct numerical simulations of the MG. Figure 2 also shows that, even if only approximate, Eq. (34) provides an useful estimate of the point where the crossover occurs.

It is possible to compute  $\sigma^2(\Gamma)$  to leading order in  $\Gamma \ll 1$ . The calculation is carried out in Appendix B in detail. The result is

$$\frac{\sigma^2}{N} \cong \frac{1-Q}{2} \left[ 1 + \frac{1-Q + \alpha(1-3Q)}{4\alpha} \Gamma + O(\Gamma^2) \right]. \quad (37)$$

The inset of Fig. 2 shows that this expression indeed reproduces quite accurately the small  $\Gamma$  behavior of  $\sigma^2$ . Note finally that Eq. (36) has a finite solution  $\sigma^2(\infty) = \sigma^2(0) + \Sigma(0)$  in the limit  $\Gamma \rightarrow \infty$ . Furthermore, it is easy to understand the origin of the behavior  $\sigma^2/N \sim 1/\alpha$  for Eq. (36). Because of the constraint, when  $\xi_i \xi_j$  is positive (negative) the fluctuations of  $\tanh(y_i) - m_i$  are positively (negatively) correlated with  $\tanh(y_j) - m_j$ . If we assume that  $\langle [\tanh(y_i) - m_i][\tanh(y_j) - m_j] \rangle \approx c \xi_i \xi_j$  for some constant  $c$ , we find  $\Sigma \approx c \sum_{i \neq j} \xi_i \xi_j^2$ . This leads easily to  $\Sigma/N \approx c/(4\alpha)$ , which explains the divergence of  $\sigma^2/N$  as  $\alpha \rightarrow 0$  for  $\Gamma \gg 1$ .

<sup>8</sup>The Monte Carlo procedure follows the usual basic steps: (i) A move  $y_i \rightarrow y_i + \epsilon \xi_i^\mu$  is proposed with  $\mu$  and  $\epsilon$  drawn at random, (ii) the “energy”

$$E\{y_i\} = \sum_{i=1}^N [\ln \cosh y_i - m_i y_i]$$

of the new configuration is computed, and (iii) the move is accepted with a probability equal to  $\min(1, e^{-\beta \Delta E})$ , where  $\Delta E$  is the “energy” difference.

### C. Selection of different initial conditions in the replica calculation

As discussed above, the stationary state properties of the MG in the symmetric phase depend on the initial conditions. Can the statistical mechanics approach to the MG [3,4] be extended to characterize this dependence for  $\Gamma \ll 1$ ? If this is possible, how do we expect the resulting picture to change when  $\Gamma$  increases? We shall first focus on the first question (i.e.,  $\Gamma \ll 1$ ) and then discuss the second.

Of course one can introduce the constraint on the distribution of  $y_i$  in the replica approach in a straightforward manner. This leads, however, to tedious calculations. We prefer to follow a different approach. In the  $\alpha < \alpha_c$  phase the minimum of  $H$  is degenerate, i.e.,  $H=0$  occurs on a connected set of points. Each of these points corresponds to a different set of initial conditions, as discussed above. In order to select a particular point with  $H=0$  we can add a potential  $\eta \sum_i (s_i - s_i^*)^2/2$  to the Hamiltonian  $H$ , which will favor the solutions closer to  $s_i^*$ , and then let the strength  $\eta$  of the potential go to zero. This procedure lifts the degeneracy and gives us the statistical features of the equilibrium close to  $s_i^*$ .

The nature of the stationary state changes as the asymmetry in the initial conditions changes. If we take  $s_i^* = s^*$ , the state at  $s^* = 0$  describes symmetric initial conditions and increasing  $s^* > 0$  gives asymmetric states.

The saddle point equations of the statistical mechanics approach of Ref. [3] can be reduced to two equations,

$$Q = \int_{-\infty}^{\infty} Dz s_0^2(z), \quad (38)$$

$$\chi = \frac{1+\chi}{\sqrt{\alpha(1+Q)}} \int_{-\infty}^{\infty} Dz z s_0(z), \quad (39)$$

where  $Dz = (dz/\sqrt{2\pi}) e^{-z^2/2}$ ,  $s_0(z) \in [-1, 1]$  is the value of  $s$  that minimizes

$$V_z(s) = \frac{1}{2} s^2 - \sqrt{\frac{1+Q}{\alpha}} z s + \frac{1}{2} \eta (1+\chi) (s - s^*)^2, \quad (40)$$

and  $\chi = \beta(Q - q)/\alpha$  is a “spin susceptibility.” There are two possible solutions. One with  $\chi < \infty$  finite as  $\eta \rightarrow 0$ , which describes the  $\alpha > \alpha_c$  phase. The other has  $\chi \sim 1/\eta$ , which diverges as  $\eta \rightarrow 0$ . This solution describes the  $\alpha < \alpha_c$  phase. We focus on this second solution, which can be conveniently parametrized by two parameters  $z_0$  and  $\epsilon_0$ . We find

$$s_0(z) = \begin{cases} -1 & \text{if } z \leq -z_0 - \epsilon_0 \\ \frac{z + \epsilon_0}{z_0} & \text{if } -z_0 - \epsilon_0 < z < z_0 - \epsilon_0 \\ 1 & \text{if } z \geq z_0 - \epsilon_0 \end{cases}$$

Indeed Eq. (38) gives  $Q(z_0, \epsilon_0)$  and Eq. (39), which for  $\chi \rightarrow \infty$  reads  $\sqrt{\alpha(1+Q)} = \int Dz z s_0(z)$ , then gives  $\alpha(z_0, \epsilon_0)$ .

With  $\epsilon_0 \neq 0$  one finds solutions with a nonzero “magnetization”  $M = \langle s_i \rangle$ . This quantity is particularly meaningful,



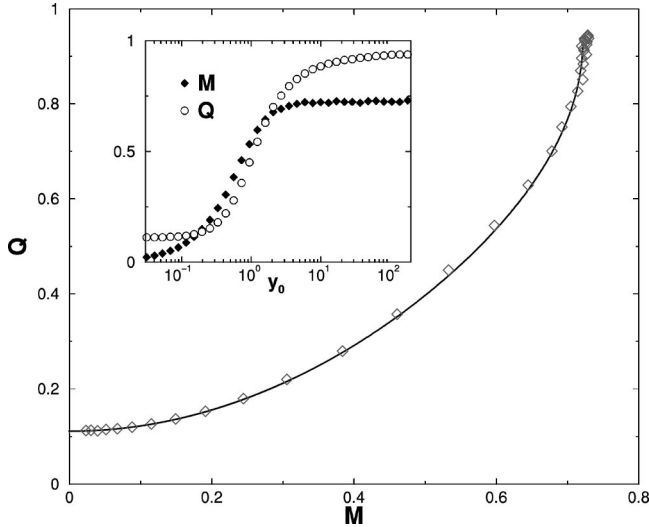


FIG. 3. Relation between  $Q$  and  $M$ , for  $\alpha=0.1$ , derived from analytic calculation (full line) and from numerical simulations of the MG with different initial conditions  $y_0$  ( $\diamond$ :  $P=32, N=320, \Gamma=0.1$ ). The inset shows the dependence of  $Q$  and  $M$  on the initial condition  $y_0$ .

in this context, because it measures the overlap of the behavior of agents in the SS with their *a priori* preferred strategies,

$$M \equiv \int_{-\infty}^{\infty} Dz s_0(z) = \lim_{t \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \langle s_i(t) s_i(0) \rangle. \quad (41)$$

Note indeed that one can always perform a ‘‘gauge’’ transformation in order to redefine  $s = +1$  as the initially preferred strategy. This amounts to taking  $y_i(0) \geq 0$  for all  $i$ .

Which SS is reached from a particular initial condition is, of course, a quite complex issue that requires the integration of the dynamics. However, the relation between  $Q$  and  $M$  derived analytically from Eqs. (38) and (41) can easily be checked by numerical simulations of the MG. Figure 3 shows that the self-overlap  $Q$  and the magnetization  $M$  computed in numerical simulations with initial conditions  $y_i(0) = y_0$  for all  $i$ , perfectly match the analytic results. The inset of this figure shows how the final magnetization  $M$  and the self-overlap  $Q$  in the SS depend on the asymmetry  $y_0$  of initial conditions.

Let us finally discuss the dependence on  $\Gamma$  for asymmetric initial conditions. Equation (34) provides a characteristic value of  $\Gamma$  as a function of  $\alpha$  and  $Q$ . This theoretical prediction is tested against numerical simulations of the MG in Fig. 4: when plotted against  $\Gamma/\Gamma_c$ , the curves of  $\sigma^2/N$  obtained from numerical simulations approximately collapse one onto the other in the large  $\Gamma$  region. Figure 4 suggests that  $\Gamma_c$  in Eq. (34) provides a close lower bound for the onset of saturation to a constant  $\sigma^2$  for large  $\Gamma$ . We find it remarkable that a formula such as Eq. (34), which is computed in the limit  $\Gamma \rightarrow 0$ , is able to predict the large  $\Gamma$  behavior.

With respect to the dependence on initial conditions, we observe that  $\Gamma_c$  is an increasing function of  $Q$  and hence it increases with the asymmetry  $y_0$  of initial conditions. Hence, for a fixed  $\Gamma$ , the fluctuation-dependent part  $\Sigma$  of  $\sigma^2$  de-

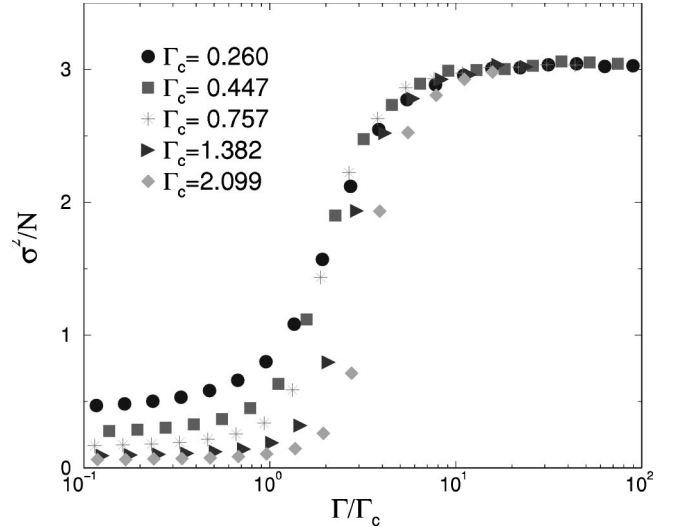


FIG. 4. Global efficiency  $\sigma^2/N$  versus  $\Gamma/\Gamma_c$  for  $\alpha=0.1 < \alpha_c$ ,  $N=160$  agents, and different initial conditions  $y_0$ . The value  $\Gamma_c$  is computed from Eq. (34) and is shown in the legend.

creases with  $y_0$  because it is an increasing function of  $\Gamma/\Gamma_c$ . This effect adds up to the decrease of the  $\Gamma$ -independent part of  $\sigma^2$  discussed previously.

Figure 4 also shows that the  $\Gamma \gg \Gamma_c$  state is independent of initial conditions  $y_0$ . This can naively be understood observing that stochastic fluctuations induce fluctuations  $\delta y_i$  that increase with  $\Gamma$ . For  $\Gamma \gg 1$  the asymmetry  $y_0$  of initial conditions is small compared to stochastic fluctuations  $\delta y_i$  and hence the system behaves as if  $y_0 \approx 0$ .

#### D. The maximally magnetized stationary state

The maximally magnetized SS (MMSS), obtained in the limit  $y_0 \rightarrow \infty$ , is also the one with the largest value of  $Q$ , and hence with the smallest value of  $\sigma^2 = N(1-Q)/2$ .  $\sigma^2/N$  is plotted against  $\alpha$  both for symmetric  $y_0=0$  initial conditions and for maximally asymmetric ones  $y_0 \rightarrow \infty$  in Fig. 5. The inset shows the behavior of  $Q$  and  $M$  in the MMSS.

Remarkably we find that  $\sigma^2/N$  vanishes linearly with  $\alpha$  in the MMSS.<sup>9</sup> This means that, at fixed  $P$ , as  $N$  increases the fluctuation  $\sigma^2$  remains constant. This is in contrast with what happens in the  $y_0=0$  state for  $\Gamma \ll \Gamma_c$ , where  $\sigma^2$  increases linearly with  $N$ , and with the case  $\Gamma \gg \Gamma_c$ , where  $\sigma^2 \propto N^2$  [18,6]. Note also that the lowest curve of Fig. 5 also gives an upper bound to the  $\sigma^2$  of Nash equilibria (see Refs. [3,4,29]).

The MMSS is also the most stable state against crowd effects: if we put  $Q(\alpha, y_0 = \infty) \cong 1 - c\alpha$ , as appropriate for the MMSS, we find that  $\Gamma_c \sim 1/\alpha$  diverges with  $\alpha$ .

## VI. CONCLUSIONS

We have clarified the correct derivation of continuous time dynamics for the MG. This on the one hand reconciles the two current approaches [3,9]. On the other it leads to a

<sup>9</sup>This result was also found analytically in [7]

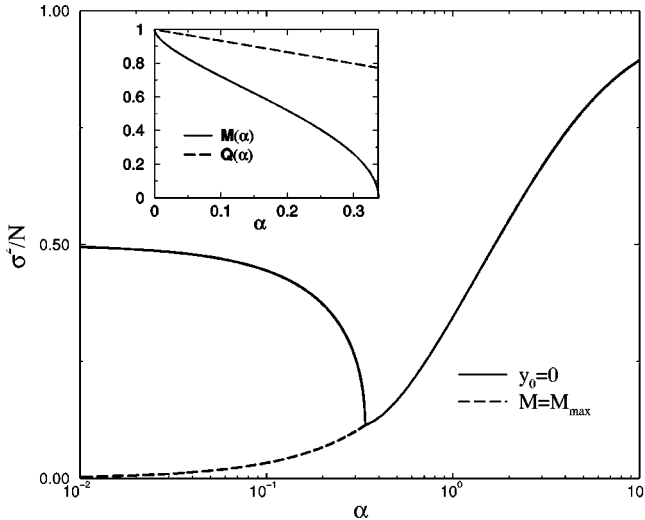


FIG. 5.  $\sigma^2/N$  for the MG with initial conditions  $y_0=0$  (full line) and  $y_0 \rightarrow \infty$  (dashed line). The inset reports the behavior of  $M$  and  $Q$  in the  $y_0 \rightarrow \infty$  SS. Note that  $Q$  is linear in  $\alpha$ .

complete understanding of the collective behavior of the MG. We confirm that stationary states are characterized by the minimum of a Hamiltonian, which measures the predictability of the game. For  $\alpha > \alpha_c$  we find a complete analytic solution, whereas for  $\alpha < \alpha_c$  the statistical mechanics approach of Ref. [3] is valid for  $\Gamma \rightarrow 0$ . It is, in principle, possible to introduce the new elements discussed here in the approach of Ref. [3] and to derive a full analytic solution. We have indeed derived the first term of the series expansion for  $\Gamma \ll 1$ , which agrees perfectly with numerical data. The extension of the approach of Ref. [3] involves lengthy calculations and it shall be pursued elsewhere.

Finally we note that the results derived in this paper generalize to more complex models. It is worth remarking that the solution to the FP equation is no more factorizable, in general, when agents account for their market impact as in Refs. [3,6,4]. Hence, as long as there are unfrozen agents, we expect that the stationary state depends on  $\Gamma$ . However, when the agents take fully into account their market impact, all of them are frozen and the conclusion that agents converge to Nash equilibria remains valid.

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#### APPENDIX A: NONLINEAR MINORITY GAMES

Take a generic dynamics

$$U_{s,i}(t+1) = U_{s,i}(t) - a_{s,i}^{\mu(t)} g[A(t)],$$

where  $g(x)$  is some function. When we carry out the limit to continuous time we find a deterministic term, which is proportional to  $-\overline{a_{s,i} \langle g(A) \rangle}$ . The stationary state conditions then read

$$v_i = -\overline{a_{s,i} \langle g(A) \rangle} \quad \text{if } f_{s,i} > 0$$

and

$$v_i > -\overline{a_{s,i} \langle g(A) \rangle} \quad \text{if } f_{s,i} = 0.$$

For any fixed  $\mu$ ,  $A(t)$  is well approximated by a Gaussian variable with mean

$$\langle A | \mu \rangle = \sum_{i,s} f_{s,i} a_{s,i}^{\mu}$$

and variance  $D = \sigma^2 - H$ . Here we neglect dependences on  $\mu$ . Also we treat  $D$  as a parameter and neglect its dependence on the stationary state probabilities  $f_{s,i}$ . Hence we can write

$$\langle g(A) | \mu \rangle = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-x^2/2} g(\langle A | \mu \rangle + \sqrt{D}x).$$

The stationary state conditions of the dynamics above can again be written as a minimization problem of the functional

$$H_g = \frac{1}{P} \sum_{\mu=1}^P \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-x^2/2} G \left( \sum_{i,s} f_{s,i} a_{s,i}^{\mu} + \sqrt{D}x \right)$$

with

$$g(x) = \frac{dG(x)}{dx}$$

and  $D = \sigma^2 - H$ , which must be determined self-consistently.

Indeed taking the derivative of  $H_g$  with respect to  $f_{s,i}$  and imposing the constraint  $f_{s,i} \geq 0$  and normalization, we arrive at exactly the same equations that describe the stationary state of the process.

The Hamiltonian for the original MG is derived setting  $g(x) = \text{sgn } x$ , which leads to

$$H_{\text{sign}} = \frac{1}{P} \sum_{\mu=1}^P \left[ \frac{1}{\sqrt{\pi}} e^{-\langle A | \mu \rangle^2 / D} + \frac{\langle A | \mu \rangle}{\sqrt{D}} \text{erf} \left( \frac{\langle A | \mu \rangle}{\sqrt{D}} \right) \right].$$

The analysis of stochastic fluctuations can be extended to nonlinear cases in a straightforward manner. Again the key point is that the dynamics is constrained to the linear space spanned by the vectors  $|a^{\mu}\rangle$ . For  $\alpha > \alpha_c$  we have no dependence on initial conditions. However, it is not easy to show, in general, that the distribution of scores factorizes across agents. This means that there may be a contribution of fluctuations to  $\sigma^2$ —i.e.,  $\Sigma > 0$ —so we cannot rule out a dependence of  $\sigma^2$  on  $\Gamma$ . Numerical simulations for  $g(x) = \text{sgn } x$  show that such a dependence, if it exists, is very weak. Anyway even though  $\sigma^2$  only depends on  $f_{s,i}$ , the minimization problem depends on  $D = \sigma^2 - H$ , which must then be determined self-consistently.

For  $\alpha < \alpha_c$  the dependence on initial conditions induces a correlation of scores across agents. As a result  $\sigma^2$  depends on  $\Gamma$  just as in the linear case discussed above.

APPENDIX B: SMALL  $\Gamma$  EXPANSION

For  $\Gamma \ll 1$  it is appropriate to consider  $\beta \gg 1$  and to take

$$y_i = \text{arc tanh } m_i + \frac{z_i}{\sqrt{\beta}}$$

so that  $\beta[\ln \cosh y_i - m_i y_i] \approx \frac{1}{2}(1 - m_i^2)z_i^2 + O(\beta^{-1/2})$ . Hence we have to sample a distribution

$$P\{z_i\} \propto \exp\left[-\frac{1}{2} \sum_i (1 - m_i^2)z_i^2\right],$$

where  $z_i$  has the form

$$z_i = \sum_{\mu=1}^P c^\mu \xi_i^\mu.$$

It is convenient to express everything in terms of the coefficients  $c^\mu$ . Their pair distribution function (PDF) is derived from that of  $z_i$  and it reads

$$P\{c^\mu\} \propto \exp\left[-\frac{1}{2} \sum_{\mu,\nu} c^\mu T^{\mu,\nu} c^\nu\right],$$

$$T^{\mu,\nu} = \sum_{i=1}^N (1 - m_i^2) \xi_i^\mu \xi_i^\nu.$$

From this we find  $\langle c^\mu c^\nu \rangle = [T^{-1}]^{\mu,\nu}$ .

Now we split the term  $\Sigma(\beta)$  into two contributions,

$$\Sigma(\beta) = \left\langle \left[ \sum_{i=1}^N \xi_i(\tanh y_i - m_i) \right]^2 \right\rangle + \sum_{i=1}^N \overline{\xi_i^2} [m_i^2 - (\tanh y_i)^2],$$

and work them out separately. For the first we use

$$\sum_{i=1}^N \xi_i^\mu (\tanh y_i - m_i) = \frac{1}{\sqrt{\beta}} \sum_{\nu} T^{\mu,\nu} c^\nu$$

so that

$$\left\langle \left[ \sum_{i=1}^N \xi_i(\tanh y_i - m_i) \right]^2 \right\rangle = \frac{1}{\beta P} \sum_{\mu,\nu,\gamma} T^{\mu,\nu} T^{\mu,\gamma} \langle c^\nu c^\gamma \rangle$$

$$= \frac{\text{Tr } T}{\beta P} \cong \frac{1-Q}{2\beta} N.$$

Within the approximation  $(1 - 3m_i^2) \approx (1 - 3Q)$  we are able to derive a closed expression also for the second term,

$$\sum_{i=1}^N \overline{\xi_i^2} [m_i^2 - (\tanh y_i)^2] = \frac{1}{\beta} \sum_{i=1}^N \overline{\xi_i^2} (1 - m_i^2) (1 - 3m_i^2) \langle z_i^2 \rangle$$

$$\approx \frac{\alpha}{\beta} \frac{1-3Q}{2} N.$$

Hence we find

$$\Sigma(\beta) \cong \left[ \frac{1-Q}{2} + \frac{1-3Q}{2} \alpha \right] \frac{1}{\beta} + O(\beta^{-2}).$$

This equation and Eq. (27) lead to Eq. (37).

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